The Inverse Kinematic Problem in Anisotropic Media

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Summary. We survey recent results on the inverse kinematic problem arising in geophysics. The question is whether one can determine the sound speed (index of refraction) of a medium by measuring the travel times of the corresponding ray paths. We emphasise the anisotropic case.

1 Introduction

The inverse kinematic problem can be described as follows An acoustic object Ω in 3D is probed for the wave speed $c(x)$, $x = (x_1, x_2, x_3) \in \Omega$, by measuring travel times (times-of-flight) between points on the boundary Γ of Ω. A travel time function $T(x, y)$ with source in $y \in \Gamma$ is related to the wave speed via the eikonal equation $c(x)|\nabla_x T(x, y)| = 1$. One asks to recover c from measurements $T(x, y)$ made for all $x, y \in \Gamma$. Travel time inversion is also an essential part of ultrasound techniques in medicine and mechanics (non-destructive evaluation).

From a geometrical point of view T measures the distance between two points with respect to the isotropic Riemannian metric $ds^2 = c(x)^{-2} dx^2$. Therefore $d(x, y) = T(x, y)$ for $x, y \in \Gamma$ is also called the boundary distance function of the metric. One asks if the boundary distance function uniquely determines the metric and therefore also c. The case considered above corresponds to an isotropic medium. In several physical examples the index of refraction is anisotropic, i.e. the wave speeds depend on direction, arises in several physical situations. One example is when an elastic medium has residual stress [8]. Another example arises in geophysics. It has been realized rather recently [2], by measuring the travel times of seismic waves, that the inner core of the Earth might exhibit anisotropic behavior, that is the speed of waves depends also on direction there with the fast direction parallel to the Earth's spin axis. I an anisotropic medium we model the wave speed as given by a symmetric, positive definite matrix $g = (g_{ij})(x)$, that is, a Riemannian metric in mathematical terms. The problem is to determine the Riemannian metric from the lengths of geodesics (ray paths) joining points in the boundary which we will denote now by $d_g(x, y)$, to make clear the dependence on the metric g.

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In this paper we will describe recent progress on the problem of determining g from d_g. In differential geometry this inverse problem has been studied because of rigidity questions and is known as the boundary rigidity problem. We review the boundary rigidity problem in Section 2 and the linearized problem in Section 3 which involves the problem of inverting the geodesic X-ray transform. In Section 4 we describe a recent local results for the boundary rigidity problem, that is, when a Riemannian metric is assumed to be close to a given one.

2 The boundary rigidity problem

Rigidity problems in differential geometry can be briefly formulated as follows: to what extent is the local geometry of a Riemannian manifold determined if some global properties are known? In particular, are two manifolds isometric under the assumption that the corresponding global properties are the same? In the last case the manifold is said to be rigid with respect to the corresponding global property. The boundary rigidity problem can be stated as to what extent is a Riemannian metric on a compact manifold with boundary determined from the distances between boundary points. We give below a more precise mathematical formulation. In what follows we consider general Riemannian manifolds with boundary but one can have in mind, as the main example the case of domains in Euclidean space in 2d or 3D equipped with a Riemannian metric.

Let (M, g) be a compact Riemannian manifold with boundary ∂M, and g' be another Riemannian metric on M. We say that g and g' have the same *boundary distance-function* if $d_g(x,y) = d_{g'}(x,y)$ for arbitrary boundary points $x, y \in \partial M$, where d_g (resp. $d_{g'}$) represents distance in M with respect to g (resp. g'). It is easy to give examples of pairs of metrics with the same boundary distance-function. Namely, if $\varphi : M \to M$ is an arbitrary diffeomorphism of M onto itself which is the identity on the boundary, then the metrics g and $g' = \varphi^* g$ have the same boundary distance-function. We say that a compact Riemannian manifold is *boundary rigid* if this is the only type of nonuniqueness. Many examples can be given of manifolds that are not boundary rigid. For instance, if an inner point $x_0 \in M \setminus \partial M$ is such that $\operatorname{dist}(x_0, \partial M) > \sup_{x,y \in \partial M} d_g(x,y)$, then we can change the metric g in a neighborhood of x_0 without changing the boundary distance-function. Another example is the hemisphere provided with the standard metric. These examples show that the boundary rigidity problem should be considered under some restrictions on the geometry of geodesics. The most usual of such restrictions is simplicity of the metric. A Riemannian manifold (M, g) (or the metric g) is called *simple* if the boundary ∂M is strictly convex and any two points $x, y \in M$ are joined by a unique geodesic. The natural conjecture is that every simple manifold is boundary rigid. The problem in this generality was proposed by Michel [M]. Simple Riemannian manifolds with boundary